B. Sc. Part III (Hons.)

7^{th} Paper $\boxed{\text{Diff. Eqns}}$ (Contd.)

Second order Equations with variable coefficients (SOEVC) (Contd)

Q. Solve $(3-x)\frac{d^2y}{dx^2} - (9-4x)\frac{dy}{dx} + (6-3x)y = 0$

Soln Given that

$$(3-x)\frac{d^2y}{dx^2} - (9-4x)\frac{dy}{dx} + (6-3x)y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - \left(\frac{9-4x}{3-x}\right)\frac{dy}{dx} + \left(\frac{6-3x}{3-x}\right)y = 0 \quad \text{---(1)}$$

It is of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{---(2)}$$

comparing (1) and (2), we get

$$P = -\frac{9-4x}{3-x} = \frac{4x-9}{3-x}, \quad Q = \frac{6-3x}{3-x}, \quad R = 0$$

Now, $1+P+Q = 1 + \frac{4x-9}{3-x} + \frac{6-3x}{3-x}$

$$= \frac{\cancel{3-x} + \cancel{4x} - \cancel{9} + \cancel{6} - \cancel{3x}}{3-x} = 0$$

$$\Rightarrow 1+P+Q = 0$$

$\Rightarrow$ e^{x} is a part of CF of eq (1). —(3)

Let $u = e^{x}$.

Let $y = uv$ be the complete soln.

$$\left.\begin{aligned} \Rightarrow \frac{dy}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx} \\ \Rightarrow \frac{d^2y}{dx^2} &= v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2} \end{aligned}\right] \text{—(4)}$$

Putting the values of (4) in (2), we get

$$\left(v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}\right) + P\left(v\frac{du}{dx} + u\frac{dv}{dx}\right) + Quv = R$$

$$\Rightarrow v\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) + u\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right) + 2\frac{du}{dx}\frac{dv}{dx} = R \quad \text{——— (5)}$$

But $R = 0$, $\Rightarrow$ $y = u$ is a soln of (2)

so it will satisfy (2).

$$\Rightarrow \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0$$

Putting this value in (5), we get

$$\Rightarrow u\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right) + 2\frac{du}{dx}\frac{dv}{dx} = R$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

Putting the values of P and Q, we have.

$$\frac{d^2v}{dx^2} + \left(\frac{4x-9}{3-x} + \frac{2}{e^x}\frac{d(e^x)}{dx}\right)\frac{dv}{dx} = 0$$

$[\because u = e^x]$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(\frac{4x-9}{3-x} + \frac{2}{e^x}\cdot e^x\right)\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(\frac{4x-9}{3-x} + 2\right)\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(\frac{4x-9+6-2x}{3-x}\right)\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \frac{2x-3}{3-x}\frac{dv}{dx} = 0 \quad \text{---} (5)$$

Put $\frac{dv}{dx} = z \Rightarrow \frac{d^2v}{dx^2} = \frac{dz}{dx}$

$$\Rightarrow \frac{dz}{dx} + \frac{2x-3}{3-x} z = 0$$

$$\Rightarrow \frac{dz}{z} + \frac{2x-3}{3-x} dx = 0$$

$$\Rightarrow \frac{dz}{z} = \frac{2x-3}{x-3} dx \Rightarrow \frac{dz}{z} = \frac{2(x-3)+3}{x-3} dx$$

$$\Rightarrow \frac{dz}{z} = \left[2 + \frac{3}{x-3}\right] dx$$

Integrating, we get

$$\log z = 2x + 3\log(x-3) + \log c$$

$$\Rightarrow z = c\, e^{2x} \cdot (x-3)^3 \quad \text{But } z = \frac{dv}{dx}$$

$$\Rightarrow \frac{dv}{dx} = c\, e^{2x} (x-3)^3$$

$$\Rightarrow dv = c \cdot e^{2x} (x-3)^3\, dx$$

Integrating we get

$$\Rightarrow v = c\int e^{2x} (x-3)^3\, dx$$

$$\Rightarrow v = c \cdot (x-3)^3 \cdot \frac{e^{2x}}{2} - c \cdot \int 3(x-2)^2 \cdot \frac{e^{2x}}{2}\, dx$$

$$\Rightarrow v = c(x-3)^3 \frac{e^{2x}}{2} - 3c\int (x-2)^2 e^{2x}\, dx$$

$$\Rightarrow v = \frac{c}{2} e^{2x} (x-3)^3 - 3c\left[(x-2)^2 \frac{e^{2x}}{2} - \int 2(x-2) \cdot \frac{e^{2x}}{\cancel{2}}\, dx\right]$$

$$\Rightarrow v = \frac{c}{2} e^{2x} (x-3)^3 - \frac{3c}{2}(x-2)^2 e^{2x} + 3c\int (x-2) e^{x}\, dx$$

$$\Rightarrow v = \frac{c}{2} e^{2x} (x-3)^3 - \frac{3c}{2}(x-2)^2 e^{2x} + 3c\left[(x-2)e^{x} - \int e^{x}\, dx\right]$$

$$\Rightarrow v = \frac{c}{2} e^{2x} (x-3)^3 - \frac{3c}{2}(x-2)^2 e^{2x} + 3c(x-2)e^{x} - 3ce^{x} + c_1$$

$\therefore$ complete soln is $y = uv$

where $u = e^{x}$ and v is given by (6).